CSM SOLUTIONS OF ROTATING BLADE DYNAMICS USING INTEGRATING MATRICES

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William D. Lakin

Department of Mathematics and Statistics
The University of Vermont
Burlington, VT 05405

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Introduction

The dynamic behavior of flexible rotating beams continues to receive considerable research attention as it constitutes a fundamental problem in applied mechanics. Further, beams comprise parts of many rotating structures of engineering significance. A topic of particular interest at the present time involves the development of techniques for obtaining the behavior in both space and time of a rotor acted upon by a simple airload loading. Most current work on problems of this type uses solution techniques based on normal modes. Even for relatively simple rotor problems, this approach can prove cumbersome and limiting. It is certainly true that normal modes cannot be disregarded, as knowledge of natural blade frequencies is always important. However, the present work has considered a computational structural mechanics (CSM) approach to rotor blade dynamics problems in which the physical properties of the rotor blade provide input for a direct numerical solution of the relevant boundary-and-initial-value problem.

Analysis of the dynamics of a given rotor system may require solution of the governing equations over a long time interval corresponding to many revolutions of the loaded flexible blade. For this reason, most of the common techniques in computational mechanics, which treat the space-time behavior concurrently, cannot be applied to the rotor dynamics problem without a large expenditure of computational resources. By contrast, the integrating matrix technique of computational mechanics has the ability to consistently incorporate boundary conditions and "remove" dependence on a space variable. For problems involving both space and time, this feature of the integrating matrix approach thus can generate a "splitting" which forms the basis of an efficient CSM method for numerical solution of rotor dynamics problems. Indeed, the resulting method, when fully developed, should be sufficiently simple that it can be run on a workstation-class computer, yet sufficiently powerful that it can be used as an engineering design tool.

Integrating matrices provide a fast and efficient means of numerically integrating a function whose values are known at increments of the independent variable, i.e. on a discrete grid with a finite number of points. When the values of the function at the grid points are arranged as a column vector, premultiplication by an integrating matrix produces a vector whose entries give approximate values of the integral of the function. An attractive feature of integrating matrices is that their derivation requires only information about the grid points, and no explicit information is needed about the function to be integrated. Thus, so long as the grid points are not changed, the same integrating matrix can be used unaltered to numerically integrate any function whose values are known at the grid points. This feature is in marked contrast to other techniques used for numerical quadrature. Separation of grid and function information also holds for differentiating matrices, which may be used to differentiate functions whose values are known on discrete grids.

Integrating matrices were originally developed for use on grids with uniformly spaced grid points [1, 2], and proved quite succesful in analyzing the vibration characteristics of rotating cantilevered beams [3, 4]. Subsequent removal of the

restriction to uniform grids [5] allowed more realistic consideration of beams with variable, or even discontinuous, physical properties. The ability to use grids with arbitrarily spaced points was found to be vital in the case of differentiating matrices, as the addition of "near boundary" points proved necessary to prevent a severe degredation in accuracy when derivatives are approximated [6].

The fact that integrating matrices separate grid point information and function values has implications for the numerical solution of differential and integro-differential equations. Consider, for example, an equation in a single space variable. By expressing the governing equation in matrix notation and utilizing integrating and differentiating matrices as integral and differential operators, dependence on the independent variable can be removed in favor of a matrix equation for the dependent variable, which can be solved by standard techniques. A key step in this approach is a reformulation of the original boundary value problem to consistently incorporate all boundary conditions. In general, this reformulation will convert a differential equation to an integro-differential equation. For rotating beam vibration problems involving cantilevered boundary conditions, a numerical solution can be achieved using only integrating matrices. However, for other types of boundary conditions, solutions may require the use of both integrating and differentiating matrices, as well as matrices which perform boundary evaluations. Problems which can be treated by this CSM method are now not limited to a single space variable. Integrating and differentiating matrices have been used to solve problems with two space dimensions [7], as well to treat beam problems including the effects of concentrated masses and other point forcings [8].

To provide the basis of a CSM method for analysis of a rotor acted upon by an airload loading, the integrating matrix technique must be extended to the space-time domain, and adapted for boundary-and-initial-value problems. A temptation exists to treat time as merely a second variable, and attempt to apply the two-variable matrices developed in [7]. However, closer examination shows this to be inappropriate in the present context. The dimensions of the two-variable matrices will depend on the number of time steps needed to reach the desired terminal value of time. In most cases, it can be expected that large time steps will not produce the desired accuracy. Consequently, the matrices in the currently available integrating matrix approach will rapidly become so large as to be impractical in a computational sense. Excessive size limitations of other CSM methods for this problem would thus not be avoided.

A more natural approach, with the potential to be highly efficient, involves making use of the ability of the integrating matrix method to "eliminate" an independent variable, replacing integrals and derivatives with respect to this variable by equivalent matrix operators. The strategy adopted in the present research has been to use the properties of single-variable space integrating and differentiating matrices (which depend only on the distribution of grid points along the rotor and not on the physical properties of the rotor) to consistently resolve the spatial dependence in the problem. The rotor dynamics will then governed by a second-order

matrix ordinary differential equation with time as the sole variable. As the initial displacements and velocities of the rotor are known at the spatial grid points, the behavior of the rotor can now be obtained through solution of an equivalent matrix initial value problem without the need to assume normal modes in space or time.

The Flap-Bending Problem

To develop an integrating and differentiating matrix approach for space-time problems in rotor dynamics, and to demonstrate the validity and efficiency of this approach, the flap bending of a rotor subject to a simple airload loading was considered. This problem is governed by the flap equation, a fourth-order partial differential equation involving space and time derivatives of the displacement function, together with pinned-free boundary conditions and initial conditions on the displacement and velocity of the rotor.

For a simple pinned-free uniform beam of length R, for which there is a flap pin at the center of rotation ($r = 0$) and both EI and the mass m are constant, the moment equation can be differentiated twice to give the loading equation [9]

$$(EI\ w'')'' - \left(T(r)w'\right)' + m\ddot{w} = L'(r,t), \tag{1}$$

$$' = \frac{\partial}{\partial r}, \quad \cdot = \frac{\partial}{\partial t},$$

where $w(r,t)$ is the transverse displacement at position r along the rotor at time t, Ω is the constant angular velocity of the rotor, and $L'(r)$ is the airload term, and

$$T(r) = \frac{m\Omega^2}{2}(R^2 - r^2). \tag{2}$$

Associated boundary conditions are

$$w(0,t) = w''(0,t) = w''(R,t) = w'''(R,t) = 0, \tag{3}$$

and the rotor dynamics problem is completed by specifying initial displacement and velocity functions for the rotor at time $t = 0$.

With the assumptions implicit in [9], the airload term can be written as

$$L'(r,t) = \frac{\gamma}{6R^2}[r + \mu R \sin \Omega t]^2 \alpha \tag{4}$$

where α is the blade element angle of attack, μ is the advance ratio, and γ is the blade Lock number. Substitution of an appropriate relation for α now gives

$$L'(r,t) = \frac{\gamma}{6R^2}\left[(r+\mu R\sin\Omega t)^2\theta(r,t) + R(r+\mu R\sin\Omega t)\lambda_s\right]$$

$$-\frac{\gamma}{6R^2}\left[\mu R\cos\Omega t(r+\mu R\sin\Omega t)w' + \frac{1}{\Omega}(r+\mu R\sin\Omega t)\dot{w}\right] \tag{5}$$

where λ_s is an inflow term and

$$\theta(r,t) = \theta_o + rR\theta_t - B_{1c}\sin\Omega t - A_{1c}\cos\Omega t \tag{6}$$

On rearangement, equation (1) now gives as the fourth-order partial differential equation which governs the simple rotor blade acted upon by simple airload loading

$$(EI\ w'')'' - \left(T(r)w'\right)' + a(r,t)w' + b(r,t)\dot{w} + m\ddot{w} = \frac{\gamma}{6R^2}f(r,t) \tag{7}$$

This equation, together with the boundary conditions (3), was taken as the starting point for the present extention of the integrating and differentiating matrix method of CSM to rotor problems involving both space and time.

Reduction to a Matrix Initial Value Problem

As a first step toward consistently "splitting" the space and time aspects of this problem, equation (7) must be reformulated and *all* boundary conditions explicitly incorporated. If the rotor were clamped at its root, rather than pinned, the two boundary conditions at $r = 0$ could be used immediately to express w and w' in terms of the "fundamental" variable w''. Two integrations of the governing equation from r to R then would serve to incorporate the boundary conditions at the rotor's free end. The result would be a reformulation in terms of the fundamental variable w'' which requires only the use of integrating matrices. From this point of view, clamped-free boundary conditions are the "natural" conditions for the CSM approach to rotor problems. By contrast, with pinned-free boundary conditions, straightforward reduction to a formulation involving only the fundamental variable w'' is *not* possible. In this sense, the present boundary conditions are not natural in the rotor context and require special treatment to achieve a consistent reformulation.

For a consistent reformulation in the present pinned-free case, equation (7) was first integrated from r to R. Integration by parts in space was applied, as required, and use was made of the fact that $T(R) = 0$, and the boundary condition that

the third space derivative of $w(r,t)$ vanishes when $r = R$ is applied. A second integration from r to R was then done, again using integration by parts as needed, as well as using the second boundary condition that $w''(r,t)$ vanishes at $r = R$.

The result of this reformulation is an integro-differential equation with time-dependent coefficients involving space integrals of $w(r,t)$, its second space derivative, and its first and second time derivatives. An integrated term involving the unknown displacement $w(R,t)$ is also present. However, neither of the boundary conditions at $r = 0$ have yet been used.

To incorporate both remaining boundary conditions, the integro-differential equation was evaluated at rotor's pinned end. The resulting consistency condition can now be substituted back into the integro-differential equation in such a way that consistency is retained in the limit as μ tends to zero. As a final prelude to derivation of an equivalent matrix problem involving only time derivatives, the various integrals, and the second space derivative in the reformulated integro-differential equation were considered as operators, and the entire expression recast into operator form.

To achieve the desired splitting in space and time, the space interval $[0, R]$ was partitioned into N subintervals through definition of a grid containing $N+1$ discrete points (including the boundary points $r = 0$ and $r = R$). This grid need not be uniform. Indeed, as the eventual problem will involve a differentiating matrix which approximates a second space derivative, a uniform grid would be inappropriate in this context. This consideration, plus the need to deal with matrices which are formally singular, required the addition of two "near boundary" grid points near each of the endpoints.

If w now denotes an $(N+1)$-dimensional column vector with $k-th$ element $w_k(t) = w(r_k, t)$, evaluating the operator equation at the grid points produces the $(N+1)$-by-$(N+1)$ matrix equation

$$G\ddot{w} + H\dot{w} + Kw = Mf \tag{8}$$

The inhomogeneous term f in this equation comes from the displacement-independent portion of the airload loading. Initial position and velocity vectors are now obtained from the original initial conditions, completing the statement of the equivalent matrix initial value problem.

The matrices in (8) involve both the 0-to-r and r-to-R integrating matrices, the second space derivative matrix, grid information matrices, a boundary evaluation matrix, and time dependent scalars associated with the airload. It is worth explicitly noting that the present context is the first reformulation in which the use of *both* integrating and differentiating matrices appears to be required.

To solve equation (8) numerically using standard techniques, G would ordinarily be inverted to obtain an explicit equation for the second time derivative. However, because of the properties of the integrating matrix, G is singular. The source of this

difficulty can be traced to the fact that in the reformulation, consistent incorporation of the boundary conditions leads to a "redundant" ($0 = 0$) equation at a boundary point, and hence a zero row in the matrix G. Indeed, this is almost a mark of the consistent reformulation. Unfortunately, this boundary point cannot simply be removed from the vector w, as this would, in turn, eliminate the influence of the boundary conditions on interior points.

Regularization of the matrix G in the present context is a matter of some delicacy involving examination of the limiting behavior of the solution as r tends to zero. Consistent treatment of the approximation to the second space derivative also requires an appropriate splitting of the matrix coefficient K in equation (8).

One of the aims of the present research program was to develop a solution package which could be implemented on a computer workstation. Use of a mainframe computer will thus not be a requirement when the techniques being developed in this work are transferred to a design and analysis context. However, while a workstation implementation will facilitate ease of use, it also introduces the need to closely monitor the accuracy of the computed solution. In a mainframe environment which allows use of 64-bit arithmetic, single precision is usually sufficient to achieve the desired accuracy in all but the largest calculations. By contrast, the current work has determined that single precision is not sufficient for accurate determination of the required matrices in the present context. Rather, double precision arithmetic must be employed in all calculations.

A Fortran code has been written and validated which, given the spatial grid points, derives the various matrices which appear as coefficients in the matrix initial value problem. All calculations are carried out using double precision on an Iris workstation.

The fact that the consistent reformulation (8) contains both integrating and differentiating matrices raises special considerations. It has already been noted that the presence of a differentiating matrix requires use of a nonuniform grid with the inclusion of near boundary points to preserve accuracy in approximations to a derivative. Accuracy is especially critical in the present case as the second space derivative is multiplied by a parameter whose magnitude leads to "stiff" behavior in the computational problem. However, placing grid points too close to a boundary can also lead to a loss of accuracy in the associated integrating matrices. Consequently, the spacing of the near boundary points must be chosen so as to achieve an acceptable balance between the need to accurately approximate integrals and derivatives. Consideration has been given to the proper spacing of these near boundary points in the nonuniform spatial grid.

A second question which involves balancing accuracies also arises when integrating and differentiating matrices must be used together. Previously, it was felt that a consistent formulation required use of the same degree polynomials in computing both the integrating and differentiating matrices associated with a given nonuniform spatial grid. However, a formal examination of the error terms in the resulting approximations has now shown that this type of "consistency" leads to much larger

errors in approximations to derivatives than to integrals. The larger errors are further exacerbated in the present context by the large magnitude of the coefficient which multiplies the second space derivative matrix. Achieving consistency in the errors associated with an integration and a differentiation has now been shown to imply that polynomials of degree $k + 2$ must be used to form a differenting matrix if polynomials of degree k are used to form the corresponding integrating matrix. Lagrange interpolation was also found to be a more robust method for obtaining the rows of the first and second space derivative matrices. The previous technique, involving least squares approximation based on polynomials which are orthogonal on the discrete grid, remains the preferred technique for the creation of integrating matrices.

While accuracy considerations are present in the calculation of the basic integrating and differentiating matrices for grids with near-boundary points, features of the equivalent time-dependent initial value problem also require great care if the computed solution is not to be contaminated by a large accumulation of numerical errors. As has been noted, because the ratio of parameters $\frac{EI}{m}$ is large, the initial value problem is numerically "stiff." In mathematical terms, this means that the Jacobian matrix of the system of differential equations has eigenvalues whose magnitudes differ greatly in size. Physically, solutions of the equations will contain components which show significant changes on highly disparate time scales. To retain accuracy using standard numerical techniques, the time step used must be that associated with the fastest time scale. A calculation based on, say, Runge-Kutta methods, will thus be both highly laborious and inefficient, requiring perhaps thousands of time steps per rotor revolution. Stiff systems require special techniques for their numerical solution. To solve the present initial value system, a version of Gear's method for stiff differential equations was implemented on an Iris workstaion.

Spurious oscillations present in earlier solution attempts have now been traced to the effect of stiffness on slightly imbalanced initial conditions. In particular, in earlier solutions, the rotor was initially taken as having a polynomial function as an initial displacement, but no initial velocity. The attempt of transients in the solution to rapidly modify the initial conditions to give consistent dispacements and velocities is now thought to have produced large accelerations, which prior numerical methodology was unable to reconcile. To avoid this hazard of the stiff system, fully consistent initial conditions on both displacement and velocity (associated with a normal mode of the unforced, nonrotating pinned-free beam) were used in test calculations.

Developments During The Funded Period

Lagrange Approach to Differentiating Matrices

The previous computation of the differentiating matrices which approximate first and second space derivatives at points of the nonuniform grid used least squares polynomials constructed from a basis set of polynomials which are orthognal with

respect to summations on the discrete grid. As previously formulated, the maximum degree of the approximating polynomials was seven. Thus, eight grid points at most could be used to obtain non-zero elements in each row of the matrices. For the present purposes, this upper limit was found to be inadequate. To achieve the desired accuracy, seventh degree polynomials must be used in the construction of integrating matrices. However, to calculate the corresponding second derivative matrix with a consistent error estimate, tenth degree polynomials must be used. The present differentiating matrix routine was thus extended to higher degree.

Lagrange interpolation was found to be a highly robust method for accomplishing this extension. Whereas general Lagrange coefficients are quite difficult to integrate (and thus impractical for use in deriving integrating matrices), their derivatives evaluated at grid points have relatively simple expressions. Further, Lagrange interpolation gives rise to explicit error estimates for approximations to both first and second derivatives at grid points. However, it was also determined that use of the higher degree interpolating polynomials may require a re-examination of the appropriate positioning of near-boundary points in the grid.

Treatment of the Pinned Boundary and Regularization

When the reformulated integro-differential equation for the forced rotating beam is discretized at the grid points to produce the matrix initial value problem, the equation associated with the pinned end of the beam vanishes identically because of consistent incorporation of the boundary conditions. This leaves a system with more unknowns than equations. In previous work, to achieve a system with as many equations as unknowns (and hence unique solutions and a regularized matrix G), a boundary condition at the pinned end was explicitly invoked to delete the first column of the various coefficient matrices. This resulted in a modified N-by-N matrix system. However, it was realized in the course of this work that, while consistent in the setting of a continuous space variable, in the discretized setting on the spatial grid, this procedure impairs the ability of the matrices involved to accurately approximate the corresponding integral and differential operators in the continuous equations. The differentiating matrix is particularly affected by this deletion, as the positive effect on accuracy of including near-boundary points is largely negated.

Two implict ways of treating the pinned boundary, which do not involve deletion of columns in coefficient matrices, were explored. Both approaches regularize the problem by appending a non-redundant equation to give a modified and invertible matrix G. In the first approach, an equation setting the second time derivative of the boundary displacement to zero is appended to the system as the $(N + 1)$-st equation. Zero initial displacement and velocity conditions are also imposed at the pinned end. This has the effect of explicitly imposing the boundary condition specifying zero displacement at the pinned end. In the second approach, a consistency condition for the displacement at the free end, which emerges during the reformulation of the original partial differential equation, is appended as the equation which

completes the system. The regularized matrix G which is the result of either of these approaches can be readily inverted in double precision on the Iris workstation using the mathematical package Mathematica.

Use of a Modified Fundamental Variable

As was noted in the Introduction, pinned boundary conditions for the rotor at $r = 0$ preclude use of the natural fundamental variable w'' in the matrix problem as $w'(0,t)$ is an unknown function of time. If $w(r,t)$ itself is used as the dependent variable, the resulting reformulation requires the use of a matrix which approximates the second space derivative. This is regretable as the coefficient of this derivative in the reformulated problem prior to discretization has a magnitude which can lead to stiff behavior in the numerical solution. Accuracy in the approximation of this derivative is thus a limiting factor in calculations which use the present formulation.

The boundary condition $w(0,t) = 0$ *does* permit the dependent variable w to be naturally expressed as an integral of the modified fundamental variable w'. Indeed, if I_0 is the 0-to-r integrating matrix, then $w = I_0 w'$. Further, if the vector giving values of w' at points of the discrete grid is used as the dependent variable in equation (8), the matrix which replaces K in (8) involves an approximation to only the first, rather than second, space derivative. A gain in the accuracy of the overall approximation is thus possible. In addition, along the lines of rotor problems with a clamped end, one of the boundary conditions at $r = 0$ will now have been consistently incorporated in a "natural" manner.

Unfortunately, when this possibility was explored, it was found that the matrix which replaces G in equation (8) when the modified fundamental variable w' is used remains singular. Further, as $w'(0,t)$ is unknown and not necessarily zero, the first regularization procedure described above can no longer be attempted. However, preliminary studies carried out during the grant period indicate that use of a consistency condition as the appended equation which completes the system and produces a nonsingular coefficient matrix G remains a possibility.

A Comparison Equation Approach

Equation (8) is complicated by the fact that it includes time-dependent forcing terms, time-dependent matrix coefficients, and is not in normal form as it contains both first and second time derivatives. To provide a cleaner setting for development of the computational methodology, attention was therefore focussed on the problem consisting of the comparison equation

$$(EI\ w'')'' + m\ddot{w} = 0, \tag{9}$$

and the pinned boundary conditions (3). Although far simpler than equation (1), this comparison equation for a nonrotating unforced beam *does* maintain the major

features of the larger problem, including the inability to use w'' as a fundamental variable and the need to use both integrating and differentiating matrices in the equivalent matrix initial value problem. Further, the pinned-free beam problem involving (9) has the distinct advantage that it can be solved exactly. In particular, solutions are of the form

$$w(r,t) = A(\sin \lambda x + B \sinh \lambda x)(\sin \omega t + C \cos \omega t), \tag{10}$$

where A, B, and C are arbitrary constants, $\omega = \frac{\lambda^2}{\nu}$ with $\nu^2 = \frac{EI}{m}$ and λ is a root of the transcendental equation

$$\tan \lambda R - \tanh \lambda R = 0, \tag{11}$$

The first few positive solutions of (11) are

$$\lambda = 0.14023580, \quad 0.25244942, \quad 0.36464909$$

Additional positive values for λ are accurately represented by the asymptotic result

$$\lambda_n = \frac{\pi}{4R}(4n+1), \tag{12}$$

Thus, using (9) through (12) and appropriate initial conditions, numerical solutions for the comparison problem can be computed, verified, and the associated computational procedure validated.

The reformulated matrix initial value problem corresponding to (9) and (3) is of the form

$$G\ddot{w} - \left(\frac{EI}{m}\right) D_2 w = 0, \tag{13}$$

where G is the *same* matrix as in equation (8) and D_2 is the differentiating matrix which approximates second derivatives. Work was begun implementing a solution of the initial value problem composed of (13) and consistent initial conditions using Gear's method for stiff systems of equations. It was intended that results from the two approaches to regularizing the matrix G would be compared for feasability, efficiency and accuracy, and that Mathematica would be used to obtain the inverse matrix G^{-1}. When equation (13) is left-multiplied by G^{-1} to obtain an explicit system for the vector $\ddot{w}$,

$$\ddot{w} = \left(\frac{EI}{m}\right) G^{-1} D_2 w, \tag{14}$$

it appears that the first and last rows and columns of G^{-1} must be zeroed to enforce the boundary conditions $w''(0,t) = w''(R,t) = 0$.

The Full CSM Approach

Once the computational procedure for the comparison problem (3) and (9) was placed on a firm foundation, it was visualized that the computational methodology would be extended in a sequence of steps leading to a CSM solution technique for the full rotating forced beam equation (7). The first step in this sequence would have involved adapting the stiff initial value solver and regularization procedures to the reformulated matrix system for the unforced rotating beam equation

$$(EI\ w'')'' - \Big(T(r)w'\Big)' + m\ddot{w} = 0, \tag{15}$$

with pinned-free boundary conditions. Next, the reformulated system (8) for the forced rotating beam equation (7) would be considered in the case of hover ($\mu = 0$ but $\gamma \neq 0$). Finally, the case where both γ and μ are nonzero w be solved.

An attractive first step in solving second-order initial value problems is to define an auxilliary dependent variable $v = \dot{w}$ and write the second order equation for $\ddot{w}$ as a first order equation involving $\dot{v}$, v and w. For example, if v denotes the $(N+1)$-dimensional column vector with k-th element $v_k(t) = \dot{w}(r_k, t)$, and I is the $(N+1)$-by-$(N+1)$ identity matrix, equation (8) is replaced by the two matrix equations

$$I\dot{w} = v, \tag{16}$$

and

$$G\dot{v} = Mf - Hv - Kw, \tag{17}$$

If Y now denotes the $(2N+2)$-dimensional column vector obtained by stacking v underneath w, equations (16) and (17) can further be combined into a single $(2N+2)$-by-$(2N+2)$ system of the form

$$\hat{G}\dot{Y} = \hat{M}\hat{f} - \hat{K}Y, \tag{18}$$

In this equation, both $\hat{G}$ and $\hat{K}$ are composed of four $(N+1)$-by-$(N+1)$ blocks. The upper-left block in $\hat{G}$ is I, the upper-right block is the $(N+1)$-by-$(N+1)$ zero matrix O, and the lower-right block in $\hat{G}$ is the matrix G in (17). Similarly, the upper-left block in $\hat{K}$ is O, the upper-right block is I, and the lower-left block in $\hat{K}$ is $-K$ in (17). Because equation (8) contains a $\dot{w}$ term and is not in normal form, there is some ambiguity in the contents of the lower-left block in $\hat{G}$ and the lower-right block in $\hat{K}$. From equation (17), it would appear that the lower-left

block in $\hat{G}$ is O while the lower-right block in $\hat{K}$ is $-H$. Indeed, this is a valid possibility. However, (17) can also be written in the form

$$H\dot{w} + G\dot{v} = Mf - Kw, \tag{19}$$

in which case the contents of these two blocks are reversed to H and O. Additional possibilities include splitting the matrix H in (17) into a sum $H = -H_1 + H_2$, in which case the lower-left block of $\hat{G}$ in (18) will contain H_1 and the lower-right block of $\hat{K}$ will be H_2.

In solving the reformulated system corresponding to (8) with γ and μ both nonzero, it is be necessary to explore the possibilities detailed above for the two blocks in $\hat{G}$ and $\hat{K}$ in order to gauge the effect of each choice on the accuracy of the evolving solution. On the basis of preliminary work completed during the grant period, a choice which seems to hold particular promise is to split the matrix H so that H_1 contains the portions of this coefficient which are independent of time.

Final Remarks

Work completed to date on this project indicates that a CSM approach to rotating beam dynamics using integrating and differentiating matrices holds the promise of providing a robust solution methodology which would be of great value in an engineering design environment. Unfortunately, because of reorganization of units at Langley Research Center, the group which has supported this promising reseach in the past has now disappeared. While an attempt will be made to continue development of the present CSM approach, its eventual evolution into an engineering design tool can no longer be assured.

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